Various Commonly Knowing Whether

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Outline

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   - What is ‘Common’ ?
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2. Various Commonly Knowing Whether
   - Different Definitions
   - Implication Relations among Cw_G
   - Expressivity of the Language
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   - Following Work

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Two keywords: Coordination and Cooperation

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- Form: $E_G\phi = \bigwedge_{i \in G} K_i\phi$
- Names in epistemology: General Knowledge or Mutual knowledge
- In Hume, Without the requisite mutual knowledge, mutually beneficial social conventions would disappear.
- The concept is important but not enough since we cannot see the cooperation or exchange of information among the group.

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However, are they enough to characterize cooperation?

- $S_G\phi$ and $D_G\phi$ are even weaker than $E_G\phi$.
- Attaining $D_G\phi$ needs to collect all knowledge of every agent in group but the 'cooperation' happens between the outsider and the group instead of the agents.

Thus, these three concepts are not enough. We need a stronger concept.

What is ‘Common’ ?

Another important concept in philosophy: **Common Sense**

1. Ordinary, normal, or average understanding (without this a man is foolish or insane) or the general sense of mankind, or of a community
2. Those plain, self-evident truths or conventional wisdom that one needed no sophistication to grasp and no proof to accept precisely because they accorded so well with the basic (common sense) intellectual capacities and experiences of the whole social body.
3. We mean the widely shared and seemingly self-evident conclusions drawn from this faculty, the truisms about which all sensible people agree without argument or even discussion

What is ‘Common’ ?

What can we obtain by those definitions of Common Sense?

- Some keywords, especially occurred repeatedly in definitions:
  1. unsophisticated
  2. self-evident
  3. **widely shared**
  4. **all human**
  5. **general**
- Except words to define ‘sense’, other words serve for ‘Common’. We can conclude that ‘common’ in ‘common sense’ means ordinarily and normally exists in all humans’ faculty. And no matter who you are, you have such faculty and could make sure any other have, even the basic background information during any communication.

A Brief History of Common knowledge

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- Here is a mathematical reasoning: If I, A, am not laughable, B will think: if I, B, am not laughable, C has noting to laugh at. Since B is not sad, I, A, must be laughable.
- The reasoning needs inference about others' knowledge. i.e. A has to infer that what B will do according to B's information.

(2) Thomas. J. Schelling: Common interests (Tacit Coordination)(1960)

A man lost his wife in a department store. It is likely that each will think of some obvious place to meet, so obvious that each will be sure that the other is sure that it is 'obvious' to both of them....Not 'what I would do if I were her?' but 'what would I do if I were she wondering what she would do if she were I wondering what I would do if I were she...'

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- $Co_{A,B}x = \bigcap(all\ compositions\ of\ A\ and\ B)x$
  $= \bigcap(A, B, AB, BA, AAA, AAB, ABA, BAA, ...)x$
  $= \bigcap_{1\leq i}^{\infty}(A \cap B)^i x$

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1. You and I have reason to believe that S holds.
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   And so on *ad infinium*

A brief History of Common knowledge

(5) With amount of studies on common knowledge in 1980s, the definition we used today finally formed:

- $C_G\phi = \bigwedge_{1 \leq k}^{\infty} E_G^k \phi$
- $M, s \models C_G\phi \Leftrightarrow M, t \models \phi$ for all t that are G-reachable from s (a state t to be reachable from state s in k steps if there exist states s_0,s_1,s_2,...,s_k such that $s_0 = s$, $s_k = t$ and for all j with $0 \leq j \leq (k-1)$, there exists $i \in G$ such that $(s_j, s_{j+1}) \in R_i$)

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- Why do we need these logical-equivalent forms?
    1. Different definitions of Ew_G
    2. Kw has no distribution with $\wedge$

Different Definitions

Now we want to define what is 'Commonly knowing whether'.

- The definition of 'knowing Whether':

$$Kw_i\phi = K_i\phi \vee K_i\neg\phi$$

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- The definition of 'knowing Whether':

$$Kw_i\phi = K_i\phi \vee K_i\neg\phi$$

- Before we use *Kw* to define *Cw*, we should define *Ew* first.
  1. $Ew^1_G\phi = E_G\phi \vee E_G\neg\phi$
  2. $Ew^2_G\phi = \bigwedge_{i\in G} Kw_i\phi$
- The two definitions are not equivalent over $\mathcal{K}$-frames.

Different Definitions

Now we begin to define Cw:

(1) Inspired by $Kw_i\phi = K_i\phi \vee K_i\neg\phi$, we could define Cw_G as follows:

$$Cw_G^1\phi = C_G\phi \vee C_G\neg\phi$$

(2) Inspired by $C_G\phi = E_G\phi \wedge C_GE_G\phi$, we could define Cw_G as follows:

$$Cw_G^{21}\phi = Ew_G^1\phi \wedge C_GEw_G^1\phi$$

$$Cw_G^{22}\phi = Ew_G^2\phi \wedge C_GEw_G^2\phi$$

Different Definitions

(3) Inspired by $C_G\phi = \bigwedge_{1\leq k}^{\infty} E_G^k\phi$, we could define Cw_G as follows:

$$Cw_G^{31}\phi = Ew_G^1\phi \wedge Ew_G^1 Ew_G^1\phi \wedge Ew_G^1 Ew_G^1 Ew_G^1\phi \ldots$$

$$Cw_G^{32}\phi = Ew_G^2\phi \wedge Ew_G^2 Ew_G^2\phi \wedge Ew_G^2 Ew_G^2 Ew_G^2\phi \ldots$$

(4) Inspired by $C_G\phi = E_G\phi \wedge \bigwedge_{i\in G} C_G K_i\phi$, we could define Cw_G as follows:

$$Cw_G^4 = Ew_G^2\phi \wedge \bigwedge_{i\in G} Cw_G^1 Kw_i\phi$$

(5)* Inspired by $C_G\phi = \bigwedge_{s\in G^+} K_s\phi$, we could define Cw_G as follows:

$$Cw_G^5\phi = \bigwedge_{s\in G^+} Kw_s\phi$$

Implication Relations among Cw_G

- $\models Cw^1\phi \to Cw^2\phi, \models Cw^2\phi \to Cw^3\phi.$

Proof.

Since $\models C\phi \to CE\phi$, and $\models CE\phi \to CEw\phi$ (because $\models E\phi \to Ew\phi$), we have $\models C\phi \to CEw\phi$. Similarly, we obtain $\models C\neg\phi \to CEw\neg\phi$. It is easy to see that $\models Ew\phi \leftrightarrow Ew\neg\phi$. Therefore, $\models C\phi \vee C\neg\phi \to CEw\phi$, i.e., $\models Cw^1\phi \to Cw^2\phi$. Since $CEw\phi = Ew\phi \wedge EEw\phi \wedge \cdots$, and also $\models E\phi \to Ew\phi$, we can show that $\models CEw\phi \to Ew\phi \wedge EwEw\phi \wedge \cdots$, that is, $\models Cw^2\phi \to Cw^3\phi$. □

Implication Relations among Cw_G

- $\models Cw^2\phi \to Cw^4\phi$. As a corollary, $\models Cw^1\phi \to Cw^4\phi$

Proof.

Since $CEw^2\phi \equiv C(\bigwedge_{i\in G} Kw_i\phi) \equiv \bigwedge_{i\in G} CKw_i\phi$, whereas $\bigwedge_{i\in G} Cw^1 Kw_i\phi \equiv \bigwedge_{i\in G}(CKw_i\phi \vee C\neg Kw_i\phi)$, it is easy to see that $\models CEw^2\phi \to Cw^4\phi$.

Moreover, Since $\models Ew^1\phi \to Ew^2\phi$, $\models CEw^1\phi \to CEw^2\phi$. Thus $\models CEw^1\phi \to Cw^4\phi$.

Therefore, no matter whether Ew is Ew^1 or Ew^2, we always have $\models CEw\phi \to Cw^4\phi$, i.e., $\models Cw^2\phi \to Cw^4\phi$. □

Implication Relations among Cw_G

Other implication conclusions:

- $\models Cw^2_G\phi \to Cw^5_G\phi$
- $\not\models Cw^4_G\phi \to Cw^5_G\phi$
- $\not\models Cw^2_G\phi \to Cw^1_G\phi$
- $\not\models Cw^3_G\phi \to Cw^2_G\phi$
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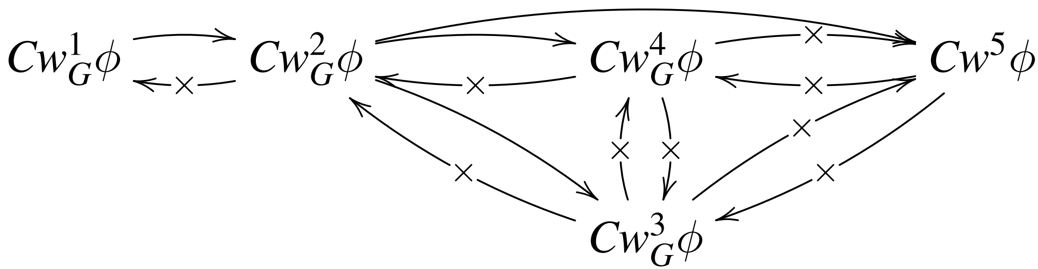
Implication Relations among Cw_G

- Thus we have the implication relations as follows:



$$over\ \mathcal{K} - frames$$

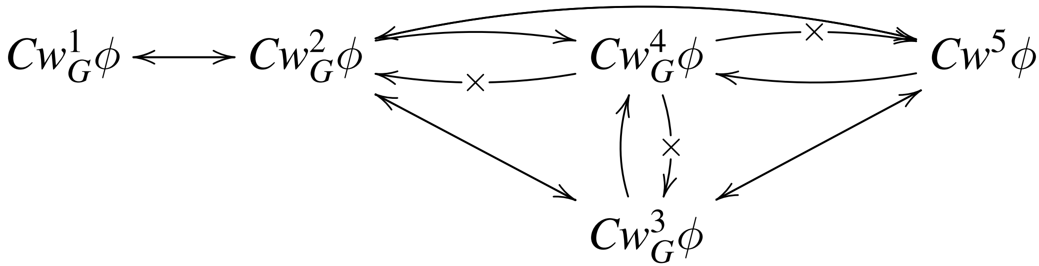
- There is neither of them are equivalent over $\mathcal{K}$-frames

Implication Relations among Cw_G

- Considering $\mathcal{T}$-frames, we have:

$$Ew^1_G\phi = Ew^2_G\phi$$

- We have the following implication relations:



$$over\ \mathcal{T} - frames$$

- Thus we have $Cw^1_G\phi \leftrightarrow Cw^2_G\phi \leftrightarrow Cw^3_G\phi \leftrightarrow Cw^5_G\phi$

Expressivity of the Language

- Over $\mathcal{T}$-frames, *PLKwCw* is equally expressive as *PLKC*.

$$
\begin{array}{lll}
t_1(p) & = & p \\
t_1(\neg\phi) & = & \neg t_1(\phi) \\
t_1(\phi \wedge \psi) & = & t_1(\phi) \wedge t_1(\psi) \\
t_1(Kw_i\phi) & = & K_i t_1(\phi) \vee K_i \neg t_1(\phi) \\
t_1(Ew\phi) & = & Et_1(\phi) \vee E\neg t_1(\phi) \\
t_1(Cw\phi) & = & Ct_1(\phi) \vee C\neg t_1(\phi)
\end{array}
\qquad
\begin{array}{lll}
t_2(p) & = & p \\
t_2(\neg\phi) & = & \neg t_2(\phi) \\
t_2(\phi \wedge \psi) & = & t_2(\phi) \wedge t_2(\psi) \\
t_2(K_i\phi) & = & Kw_i t_2(\phi) \wedge t_2(\phi) \\
t_2(E\phi) & = & Ewt_2(\phi) \wedge t_2(\phi) \\
t_2(C\phi) & = & Cwt_2(\phi) \wedge t_2(\phi)
\end{array}
$$

- Over $\mathcal{K}$-frames, $PLKwCw^1 \prec PLKC$, $PLKwCw^2 \prec PLKC$(if G is finite)

Expressivity of the Language

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Recall the concepts of distinguishing power and expressive power here.:

1. Distinguishing power: can a language tell the difference between two models.
   For any two models (M, s) and (N, r), for any $\phi \in L_1$, $M, s \models \phi$ and $N, r \models \neg\phi$ implies there exists a formula $\psi \in L_2$ such that $M, s \models \psi$ and $N, r \models \neg\psi$. We say $L_1 \preceq_d L_2$.

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2. Expressive power: which classes of models can be defined by **a formula** of the language.
   For any two classes of models $\mathcal{M}$ and $\mathcal{N}$, for any formula $\phi \in L_1$, $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\phi$ implies there exists one formula $\psi \in L_2$ such that $\mathcal{M} \models \psi$ and $\mathcal{N} \models \neg\psi$. We say $L_1 \preceq_e L_2$.

Expressivity of the Language

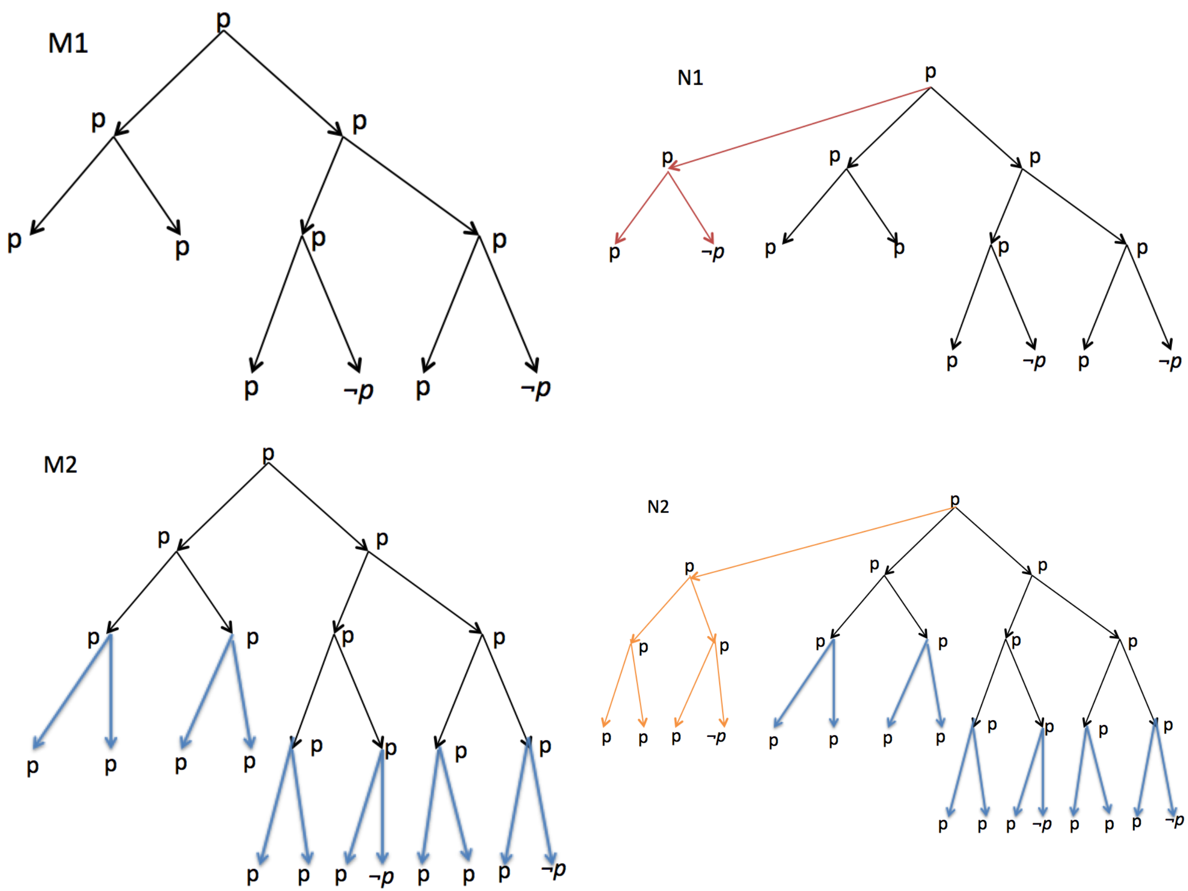
- Thus, in order to prove Cw^5 cannot be expressed by K and C, we have to find two classes of $\mathcal{K}$-models $\mathcal{M}$ and $\mathcal{N}$ such that there is some formula $\phi \in PLKwCw$, $\mathcal{M} \models \phi$ and $\mathcal{N} \not\models \phi$ but we cannot find a formula in $PLKC$ do so.

Expressivity of the Language

- Define $M = \{M_n \mid 1 \leq n\}$ and $N = \{N_n \mid 1 \leq n\}$ as follows:
  For every $n \geq 1$, $M_n = \langle W_n, R_n, V_n \rangle$ where
  $W_n = \{r, s_{011}, t_{011}\} \cup \{s_{ijk} \mid 1 \leq i \leq n, 1 \leq j \leq 2^{n-1}, k = 1, 2\} \cup \{t_{ijk} \mid 1 \leq i \leq n+1, 1 \leq j \leq 2^n, k = 1, 2\}$;
  $R_n = \{< r, s_{011} >, < r, t_{011} >\} \cup \{< s_{ijk}, s_{(i+1)(2j+k-2)k^{`}} >| s_{ijk} \in W_n, s_{(i+1)(2j+k-2)k^{`}} \in W_n, k' = 1, 2\} \cup \{< t_{ijk}, t_{(i+1)(2j+k-2)k^{`}} >| t_{ijk} \in W_n, t_{(i+1)(2j+k-2)k^{`}} \in W_n, k' = 1, 2\}$
  $V_n(p) = W_n - \{t_{(n+1)2^n 2}, t_{(n+1)2^{(n-1)}2}\}$
  Then we define $N_n = \langle W'_n, R'_n, V'_n \rangle$ from M_n:
  $W'_n = W_n \cup \{w_{011}\} \cup \{w_{ijk} \mid 1 \leq i \leq n, 1 \leq j \leq 2^{n-1}, k = 1, 2\}$
  $R'_n = R_n \cup \{< r, w_{011} >\} \cup \{< w_{ijk}, w_{(i+1)(2j+k-2)k'} >| w_{ijk} \in W'_n, w_{(i+1)(2j+k-2)k'} \in W'_n, k' = 1, 2\}$
  $V'_n(p) = W'_n - \{t_{(n+1)2^n 2}, t_{(n+1)2^{(n-1)}2}, w_{n2n-12}\}$

Expressivity of the Language

$\mathcal{M}$ and $\mathcal{N}$ are as follows:



Expressivity of the Language

- We can find that for every $M_n \in \mathcal{M}$, there is $M_n, r \models KwCw^5_G\phi$ and for every $N_n \in \mathcal{N}$, there is $N_n, r \models \neg KwCw^5_G\phi$.
- However, for every $n \geq 1$, $M_n, r \models \phi \Leftrightarrow N_n, r \models \phi$ if ϕ is $PLKwCw$-formula and $d(\phi) \leq n$. Thus, for any $PLKwCw$-formular, since its modal depth is finite, we can always find a n such that ϕ cannot distinguish (M_n, r) and (N_n, r). That means we cannot find a $PLKwCw$-formula to distinguish these two classes of models.

Problems in Axiomatization

- Axiom System of *PLKC*
  1. Axioms and Rules in K-System
  2. $C_G(\phi \to \psi) \to (C_G\phi \to C_G\psi)$
  3. $C_G\phi \to (\phi \wedge E_GC_G\phi)$ (mix)
  4. $C_G(\phi \to E_G\phi) \to (\phi \to C_G\phi)$ (induction axiom)
  5. from ϕ, infer $C_G\phi$
- However, *PLKwCw* have no such axioms.
  1. *Kw* has no K-axiom.
  2. *Cw* has no distribution.
  3. *Cw* has no mix.
  4. *Cw* has no induction.

Problems in Axiomatization

Inspired by the following Kw-axioms:

- $Kw(\chi \to \phi) \wedge Kw(\neg\chi \to \phi) \to Kw\phi$
- $Kw\phi \to kw(\phi \to \psi) \vee Kw(\neg\phi \to \chi)$

We have verified that there are

1. $Cw(\chi \to \phi) \wedge Cw(\neg\chi \to \phi) \to Cw\phi$
   if Cw is Cw^1,Cw^{21}, Cw^{22},Cw^5
2. $Cw\phi \to Cw(\phi \to \psi) \vee Cw(\neg\phi \to \chi)$ if Cw is Cw^1,$Cw^2 1$. $Cw^2 2$ and Cw^5 are not valid.
3. I do not verify the case of Cw^3 and Cw^4.

Generally, we ask for help from proof of completeness in order to finish the complete axiomatization.

Classical Proof Idea of Completeness of *PLKC*

We may 'borrow' the idea of the proof in *PLKC* in order to prove the completeness of *PLKwCw*.

- Basic proof idea: Canonical Model.
- However, *PLKC* is not impact. Consider the following set:

$$\Gamma = \{E_G^n p \mid n \in N\} \cup \{\neg C_G p\}$$

- Every finite subset of Γ is satisfiable, bu Γ is not! Thus we cannot guarantee that the union of countably many consistent sets is satisfiable.

Classical Proof Idea of Completeness of *PLKC*

We need to restrict maximal consistent set of formulas to finite maximal consistent set of formulas.

Classical Proof Idea of Completeness of *PLKC*

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- How to restrict? For every formula, we define a closure $cl(\phi)$:
  1. $\phi \in cl(\phi)$.
  2. if $\psi \in cl(\phi)$, then $sub(\psi) \in cl(\phi)$.
  3. if $\psi \in cl(\phi)$ and ψ is not negation, then $\neg\psi \in cl(\phi)$.
  4. if $C_G\phi \in cl(\phi)$, then $\{K_iC_G\psi \mid i \in G\} \subset cl(\phi)$.

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  4. if $C_G\phi \in cl(\phi)$, then $\{K_iC_G\psi \mid i \in G\} \subset cl(\phi)$.
- We can prove that the closure is finite.
- When construct the canonical model, we just use the maximal consistent set of formulas in the closure, which guarantees that all these sets can be satisfied.

Classical Proof Idea of Completeness of *PLKC*

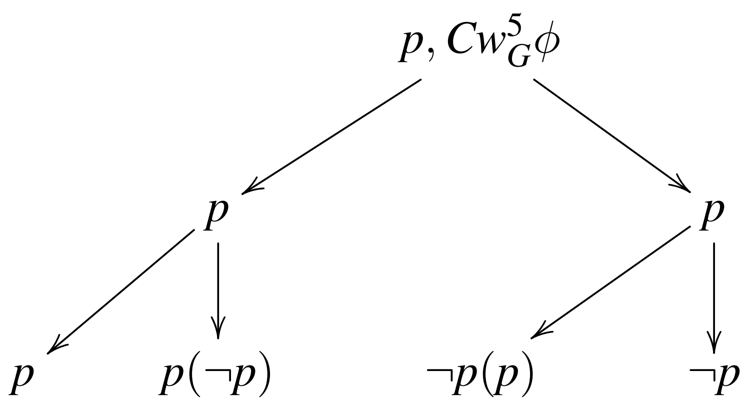
- Construction of the canonical model is as usual.
- To Prove Truth Lemma, $\phi \in \Gamma \Leftrightarrow (M^c, \Gamma) \models \phi$, we need to prove:
  - If $C_G\psi \in \Phi$, then $C_G\psi \in \Gamma$ iff every G-path from Γ is a ψ-path when the case of $\phi = C_G\psi$.
- With this lemma, we can finish the proof of Truth Lemma in help with Inductive Hypothesis.

Difficulties in proof of Completeness of *PLKwCw*

- We cannot find an obvious property of $Cw_G^5\phi$ where ϕ appears regularly. For lack of this, we cannot use Inductive Hypothesis when proving Truth Lemma.

Difficulties in proof of Completeness of *PLKwCw*

- We cannot find an obvious property of $Cw^5_G\phi$ where ϕ appears regularly. For lack of this, we cannot use Inductive Hypothesis when proving Truth Lemma.
- What we only found is, if $M, s \models Cw^5_G\phi$ where M is a complete two-branch tree, the number of nodes where ϕ is true is even on every layer of the tree.



Difficulties in proof of Completeness of *PLKwCw*

- However, we cannot guarantee that the canonical model we construct in traditional way is a two-branch tree.

1. In order to solve this problem, we hope we can find a way to transform the canonical model into a two-branch tree. But we can only unravel the canonical model into a tree model.
2. We have proved that unraveling keeps the satisfaction of every formula in *PLKwCw* in corresponding nodes: $M^c, g(s) \models \phi$ iff $M^T, s \models \phi$ for every $\phi \in PLKwCw$.

And then, we are blocked...

Following Work

1. Through proving the completeness, we hope give the complete axiomatization.
2. By the study on Cw_G^5, we hope to brighten a path to 'what is the ultimate ignorance?'.
   A tentative idea is:

$$I_G\phi =_{df} \bigwedge_{s\in G^+} \neg Kw_s\phi$$